Regular Graded Skew Clifford Algebras of Low Global Dimension

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In this talk, $\mathbb{K}$ is an algebraically closed field with $\text{char}(\mathbb{K}) \neq 2$ and $\mathbb{K}^\times$ denotes $\mathbb{K} \setminus \{0\}$.

**Definition of Graded Algebras in this talk [ATV1]:**
A $\mathbb{K}$-algebra $A$ is called a **graded algebra** if

1. $A = \bigoplus_{i \geq 0} A_i$ where $A_i$ are vector spaces over $\mathbb{K}$,
2. $\dim A_1 < \infty$,
3. $A_i A_j \subseteq A_{i+j}$, for all $i, j$
4. $A_0 = \mathbb{K}$,
5. $A$ generated by $A_1$ only.

For each $i$, $A_i$ is the span of the homogeneous elements of degree $i$.

**Examples:**
- Polynomial ring: $\mathbb{K}[x_1, \ldots, x_d]$
- Free algebra: $\mathbb{K}\langle x_1, \ldots, x_d \rangle$
Definition of Regular Algebras[ATV1]:

A graded $K$-algebra $A$ is called a **regular algebra** if

- $A$ has finite global dimension,
- $A$ has polynomial growth,
- $A$ is Gorenstein.

**Examples:**

- Polynomial ring: $\mathbb{K}[x_1, \ldots, x_d]$

- $S = \frac{\mathbb{K}\langle z_1, \ldots, z_n \rangle}{\langle z_j z_i - \mu_{ij} z_i z_j \rangle}$, $\deg(z_i) = 1$ for all $i$, $\mu_{ij} \in \mathbb{K}^\times$ and $\mu_{ij} \mu_{ji} = 1$ for all $i, j$, $\mu_{ii} = 1$ for all $i$. 
Definition of Graded Skew Clifford Algebras (GSCA): [CV]

For \( \{ i, j \} \subset \{ 1, \ldots, n \} \), let \( \mu_{ij} \in K^\times \) satisfy \( \mu_{ij} \mu_{ji} = 1 \) for all \( i \neq j \), and write \( \mu = (\mu_{ij}) \in M(n, K) \). A matrix \( M \in M(n, K) \) is called \( \mu \)-symmetric if \( M_{ij} = \mu_{ij} M_{ji} \) for all \( i, j = 1, \ldots, n \). Henceforth, suppose \( \mu_{ii} = 1 \) for all \( i \), and fix \( \mu \)-symmetric matrices \( M_1, \ldots, M_n \in M(n, K) \).

A **graded skew Clifford algebra** associated to \( \mu \) and \( M_1, \ldots, M_n \) is a graded \( K \)-algebra on degree-one generators \( x_1, \ldots, x_n \) and on degree-two generators \( y_1, \ldots, y_n \) with defining relations given by:

(a) \( x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k \) for all \( i, j = 1, \ldots, n \), and

(b) the existence of a normalizing sequence \( \{ r_1, \ldots, r_n \} \) of homogeneous elements that span \( K y_1 + \cdots + K y_n \).
Example:

Let $\mu_{12} \in K$, $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ \mu_{21} & 2 \end{bmatrix}$.

Then any graded skew Clifford algebra $A$ associated to $M_1, M_2$ satisfies

$$\frac{K \langle x_1, x_2 \rangle}{\langle x_1 x_2 + \mu_{12} x_2 x_1 - x_2^2 \rangle} \rightarrow A$$

since $x_1 x_2 + \mu_{12} x_2 x_1 = y_2$, $y_1 = x_1^2$, and $y_2 = x_2^2$. 
Definition of Quadric System:
Let $S$ be the $\mathbb{K}$-algebra on generators $z_1, \ldots, z_n$ with defining relations

$$z_j z_i = \mu_{ij} z_i z_j \quad \text{for all} \quad i, j$$

and let

$$q_k := \begin{bmatrix} z_1 & \ldots & z_n \end{bmatrix} M_k \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in S.$$ 

We say $\{q_1, \ldots, q_n\}$ is a quadric system.
Definition of Normalizing Quadric System:
A quadric system \( \{ q_1, \ldots, q_n \} \) is \textbf{normalizing} if \( \sum_{k=1}^{n} \mathbb{K}q_k \subset S \) is spanned by a normalizing sequence of \( S \).

\textbf{Example:}
For \( A \) defined as above, \( S = \frac{\mathbb{K}\langle z_1, z_2 \rangle}{\langle z_2z_1-\mu_{12}z_1z_2 \rangle} \). We also have

\[ q_1 = z_1^2 \quad q_2 = z_1z_2 + z_2^2. \]

\( \{ q_1, q_2 \} \) is a normalizing quadric system if \( \mu_{12} = -1 \).
Definition of Base-Point Free:
Let $Z$ be the zero locus in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of the defining relations of $S$, i.e.,

$$Z = \bigcap_{i,j} \mathcal{V}(z_j z_i - \mu_{ij} z_i z_j) \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}.$$ 

The quadric system $\{q_1, \ldots, q_n\}$ is **base-point free** (BPF) if $Z \cap \mathcal{V}(q_1, \ldots, q_n)$ is empty.

**Example:**
Referring to previous example, $Z = \{((a_1, a_2), (a_1, \mu_{21} a_2)) : (a_1, a_2) \in \mathbb{P}^2\}$ and $\{q_1, q_2\}$ is BPF.
Theorem [CV]: Let $\mu$ be as defined earlier, and let $M_1, \ldots, M_n$ be $\mu$-symmetric $n \times n$ matrices. A graded skew clifford algebra $A$ associated to $\mu$ and $M_1, \ldots, M_n$ is quadratic, regular of global dimension $n$ and satisfies the Cohen-Macaulay property with Hilbert series $\frac{1}{(1-t)^n}$ if and only if

the quadrics in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ determined by the $M_k$ are BPF and form a normalizing quadric system. In this case, $A$ is unique up to isomorphism, noetherian and has no zero divisors.
Quadratic regular algebras of global dimension 1 \([\text{CV}]\)

\(K[x]\) where \(\mu = 1\) and \(M_1 = 1\) is a regular graded skew Clifford algebra.

Quadratic regular algebras of global dimension 2 \([\text{CV}]\)

(i) \(\frac{K\langle x_1, x_2 \rangle}{\langle x_1 x_2 + \lambda x_2 x_1 \rangle}\) where \(\lambda \in K^\times\)

\[M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \text{ where } \mu_{12} = \lambda, \text{ is a regular graded skew Clifford algebra.}\]

(ii) \(\frac{K\langle x_1, x_2 \rangle}{\langle x_1 x_2 - x_2 x_1 - x_1^2 \rangle}\)

\[M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \text{ where } \mu_{12} = -1, \text{ is a regular graded skew Clifford algebra.}\]
(i) First Family of Examples: [N]

\[ A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle} \]

\[ g_1 = x_1 x_2 + \mu_{12} x_2 x_1 - \lambda_1 x_3^2, \quad g_2 = x_1 x_3 + \mu_{13} x_3 x_1 - \lambda_2 x_2^2, \]
\[ g_3 = x_2 x_3 + \mu_{23} x_3 x_2 - \lambda_3 x_1^2, \]

where \( \lambda_i \in \mathbb{K} \) for all \( i \).

\[ M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & \mu_{32} \lambda_3 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & \lambda_2 \\ 0 & 2 & 0 \\ \mu_{31} \lambda_2 & 0 & 0 \end{bmatrix}, \]
\[ M_3 = \begin{bmatrix} 0 & \lambda_1 & 0 \\ \mu_{21} \lambda_1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \]
For different choices $\lambda_i$ and various conditions on the $\mu_{ij}$, these algebras are regular graded skew Clifford algebras, and the point schemes that can be obtained are:

\[ P^2, \quad \triangle, \quad \bigcirc, \quad \mathbb{Z} \]
Ore Extension of graded Skew Clifford Algebras of Global Dimension 2: [N]
To figure out which of the other types of quadratic regular algebras of global dimension 3 are graded skew Clifford algebras or related to graded skew Clifford algebras, we use the notion of Ore extension.
Theorem: Every point scheme of quadratic regular algebras of global dimension 3 can be obtained from either a regular graded skew Clifford algebra of global dimension 3 or from an Ore extension of a regular graded skew Clifford algebra of global dimension 2.

These results have been extended in my paper “Classifying Quadratic Quantum $\mathbb{P}^2$s By Using Graded Skew Clifford Algebras” with M. Vancliff, and Jun Zhang [NVZ] in which we classify all quadratic regular algebras of global dimension 3 using regular graded skew Clifford algebras.
Let $B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1 x_2 - q x_2 x_1, x_1 x_3 - q^{-1} x_3 x_1, x_2 x_3 - q x_3 x_2 \rangle}$, and $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = q x_1$, $\sigma(x_2) = q x_2$, $\sigma(x_3) = q x_3$ where $q^3 \neq 1$, $q = -1, \pm i$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = x_3^2$, $\delta(x_2) = x_1^2$, and $\delta(x_3) = x_2^2$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra $A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ where

$g_1 = x_4 x_2 - q x_2 x_4 - x_1^2$, $g_2 = x_4 x_3 - q x_3 x_4 - x_2^2$,
$g_3 = x_4 x_1 - q x_1 x_4 - x_3^2$, $g_4 = x_2 x_3 - q x_3 x_2$, $g_5 = x_1 x_3 - q^{-1} x_3 x_1$,
$g_6 = x_1 x_2 - q x_2 x_1$ has point scheme given by one point. Moreover, $B$ is a regular graded skew Clifford algebra.
Let $B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1 x_2 - q^{-1} x_2 x_1, x_1 x_3 - q^{-1} x_3 x_1, x_2 x_3 - x_3 x_2 \rangle}$, and $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = x_1, \sigma(x_2) = qx_2, \sigma(x_3) = qx_3$ where $q \in \mathbb{K}^\times$, and $q^2 \neq 1$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = (q - q^{-1})x_2 x_3$, and $\delta(x_2) = 0 = \delta(x_3)$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra $A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ where $g_1 = x_4 x_2 - qx_2 x_4$, $g_2 = x_4 x_3 - qx_3 x_4$, $g_3 = x_2 x_1 - qx_1 x_2$, $g_4 = x_3 x_1 - qx_1 x_3$, $g_5 = x_2 x_3 - x_3 x_2$, $g_6 = x_4 x_1 - x_1 x_4 - (q - q^{-1})x_2 x_3$ has point scheme given by \[ \mathcal{V}(x_2, x_3) \cup \mathcal{V}(x_2 x_3 - x_1 x_4) \]. Moreover, $B$ is a regular graded skew Clifford algebra.
Let $B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle}$, and $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = x_1 - \alpha x_3$, $\sigma(x_2) = x_2$, $\sigma(x_3) = x_3$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = \alpha x_1 x_2$, and $\delta(x_2) = 0 = \delta(x_3)$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra

$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$

where $g_1 = x_1x_2 - x_2x_1$, $g_2 = x_4x_2 - x_2x_4$, $g_3 = x_3x_2 - x_2x_3$, $g_4 = x_1x_3 - x_3x_1$, $g_5 = x_4x_3 - x_3x_4$, $g_6 = x_4x_1 - x_1x_4 + \alpha(x_4x_3 - x_1x_2 - 2)$, where $\alpha \in \mathbb{K}^\times$ has point

Scheme given by $\mathcal{V}(x_1x_2 - x_3x_4)(\subset \mathcal{V}(x_2, x_3))$. Moreover, $B$ is a regular graded skew Clifford algebra.
Let $B = \frac{K\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle}$, $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = x_1$, $\sigma(x_2) = (1 + \alpha)x_2$, $\sigma(x_3) = x_3$ where $\alpha \in K^\times$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = 0 = \delta(x_3)$, and $\delta(x_2) = -\alpha x_1^2$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra $A = \frac{K\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ where $g_1 = x_1x_2 - x_2x_1$, $g_2 = x_1x_4 - x_4x_1$, $g_3 = x_1x_3 - x_3x_1$, $g_4 = x_2x_3 - x_3x_2$, $g_5 = x_4x_3 - x_3x_4$, $g_6 = x_2x_4 - x_4x_2 - \alpha(x_1^2 - x_2x_4)$ has point scheme given by $Q \cup L$ where $Q = \mathcal{V}(x_1^2 - x_2x_4)$ and $L = \mathcal{V}(x_1, x_3)$. Moreover, $B$ is a regular graded skew Clifford algebra.
Let $B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1 x_2 - x_2 x_1, x_1 x_3 - x_3 x_1, x_2 x_3 - x_3 x_2 \rangle}$, and $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = x_1 + x_3$, $\sigma(x_2) = x_2$, $\sigma(x_3) = x_3$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = -x_1^2$, and $\delta(x_2) = 0 = \delta(x_3)$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra $A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ where $g_1 = x_1 x_3 - x_3 x_1$, $g_2 = x_3 x_4 - x_4 x_3$, $g_3 = x_3 x_2 - x_2 x_3$, $g_4 = x_1 x_2 - x_2 x_1$, $g_5 = x_2 x_4 - x_4 x_2$, $g_6 = x_1 x_4 - x_4 x_1 - x_1^2 + x_4 x_3$ has point scheme given by $Q \cup L$ where $Q = \mathcal{V}(x_2^2 - x_4 x_3)$ and $L = \mathcal{V}(x_3, x_4)$ ($L$ is tangential to $Q$ at a nonsingular point of $Q$). Moreover, $B$ is a regular graded skew Clifford algebra.
Let $B = \frac{K\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle}$, and $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = x_1$, $\sigma(x_2) = x_2$, $\sigma(x_3) = x_3$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = -x_1^2 + x_2x_3$, and $\delta(x_2) = 0 = \delta(x_3)$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra $A = \frac{K\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ where $g_1 = x_1x_2 - x_2x_1$, $g_2 = x_1x_3 - x_3x_1$, $g_3 = x_2x_3 - x_3x_2$, $g_4 = x_2x_4 - x_4x_2$, $g_5 = x_3x_1 - x_4x_3$, $g_6 = x_1x_4 - x_4x_1 - x_1^2 + x_2x_3$ has point scheme given by $Q \cup L$ where $Q = \mathcal{V}(x_1^2 - x_2x_3)$ and $L = \mathcal{V}(x_2, x_3)$ ($L$ is tangential to $Q$ at a singular point of $Q$). Moreover, $B$ is a regular graded skew Clifford algebra.
Let $B = \frac{K \langle x_1, x_2, x_3 \rangle}{ \langle x_1 x_2 - x_2 x_1, x_1 x_3 - x_3 x_1, x_2 x_3 - x_3 x_2 \rangle}$, and $\sigma \in \text{Aut}(B)$ such that $\sigma(x_1) = x_1$, $\sigma(x_2) = x_2$, $\sigma(x_3) = x_3$. If $\delta$ is a left $\sigma$-derivation of $B$ such that $\delta(x_1) = 0 = \delta(x_2)$, and $\delta(x_3) = -x_1^2 + x_2 x_3$, then $A = B[x_4; \sigma, \delta]$ is a regular algebra. The algebra $A = \frac{K \langle x_1, x_2, x_3, x_4 \rangle}{ \langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ where $g_1 = x_1 x_2 - x_2 x_1$, $g_2 = x_1 x_3 - x_3 x_1$, $g_3 = x_1 x_4 - x_4 x_1$, $g_4 = x_2 x_3 - x_3 x_2$, $g_5 = x_2 x_4 - x_4 x_2$, $g_6 = x_3 x_4 - x_4 x_3 - x_1^2 + x_2 x_3$ has point scheme given by $Q \sqcup L$, where $Q = \mathcal{V}(x_1^2 - x_2 x_3)$ and $L = \mathcal{V}(x_1, x_2) \subset Q$. Moreover, $B$ is a regular graded skew Clifford algebra.


Thank You!