Introduction

History

In non-commutative algebraic geometry, we use geometric methods to understand algebra and its origins. The origins of this kind of non-commutative algebraic geometry arose in the late 1980's.

Definitions

Let $K$ be an algebraically closed field with char$(K) \neq 2$.

We will consider algebras that have the following properties:

1. $A = \bigoplus_{i,j \geq 0} A_{ij}$, where $A_i$ are vector spaces over $K$, (2) dim $A_i < \infty$, (3) $A_i A_j \subset A_{i+j}$, (4) $A_0 = K$, (5) $A$ is generated by $A_1$ only.

$A$ is called graded algebra (there exists more general definition).

A $K$-algebra $A$ is called quadratic if:

1) $A$ is graded (as defined above),
2) $A$ is a quotient of the free algebra by homogeneous relations of degree 2.

A graded $K$-algebra $A$ is called a regular algebra of global dimension $(\text{gldim}) d$ ([ATV1]) if

• $A$ has finite global dimension $d$,
• $A$ has polynomial growth,
• $A$ is Gorenstein.

Example: $S = \langle x_1, x_2, x_3 \rangle$ where $x_i$’s have degree 1, $x_i x_j = x_j x_i$, $i,j = 1,2,3$.

Quadratic regular algebras of global dimension one are isomorphic to $K[x]$.

Quadratic regular algebras of global dimension two are 2 types (up to isomorphism):

$\langle x_1, x_2 | x_3 \rangle$ where $\lambda \in K^*$ and $x_1^2 = \lambda x_1, x_2^2 = \lambda x_2 - \lambda x_1$.

The quadratic regular algebras of global dimension 3 can be described using geometry, i.e. the point variety $E \subset \mathbb{P}^2$. These algebras are used in line as well as those that are “generic” are given in [ATV1][ATV2] and entail: triangle, (triple) line, elliptic curve, conic union a line, $\mathbb{P}^2$, a union of $n$ lines where $n \in \{2;3\}$ with one intersection point.

Open Questions & Ideas

Ongoing research (open question) is classify regular algebras of global dimension 4 and subproblem is classify quadratic regular algebras of global dimension 4.

Cassidy and Vancliff defined a class of algebras that provide an “easy” way to write down some quadratic regular algebras of global dimension $d$, as follows.

Fix $\mu = (\mu_{ij}) \in M_{d}(K)$ where $\mu_{ij} \neq 0$ for all $i,j$. Say $M = (\mu_{ij}) \in M_{d}(K)$ is $\mu$-symmetric if $M = \mu M^T \mu^{-1}$, for all $i,j$.

Let $M_1, .., M_n \in M_{d}(K)$ be $\mu$-symmetric where $\mu_{ij} = 1$, $\forall i,j$. Let $A$ be a $K$-algebra on degree 1 generators $x_1, .., x_n$ and on degree 2 generators $y_1, .., y_n$ with defining relations given by

1. $x_i x_j = \mu_{ij} y_i, \forall i,j$. $y_j y_k = \sum_{i=1}^{n} (\mu_{ij} y_i) y_k$ for all $i,j$.
2. the existence of a normalizing sequence $\{r_1, .., r_n\}$ that spans $K y_i + .. + y_n$, and call $A$ a graded skew Clifford algebra $[CV]$.

Examples of Graded Skew Clifford Algebras

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Examples of Graded Skew Clifford Algebras

Let $S$ be a $K$-algebra on generators $z_1, .., z_n$ with defining relations $z_i z_j = \mu_{ij} z_j z_i$, and let

$q_i = \langle z_2, .., z_n, M_1 \rangle z_1 \in S$.

Let $Z$ be the zero locus in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of the defining relations of $S$, i.e.

$Z = \prod_{i< j} (z_i - \mu_{ij} z_j) \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Define the quadric determined by $q_i$ (or $M_1$) to be the zero locus in $Z$ of $q_i = \forall (i,j) \in Z \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

$q_i, .., q_n$ is a normalizing quadric system if $\sum_{i=1}^{n} K q_i \subset S$ is spanned by a normalizing sequence of $S$, that is contained in $S_{2}$.

Question: If $d = 4$, how much dose this [CV] class of algebras help the classification problem?


The graded skew Clifford algebra $A$ is (unique up to isomorphism) quadratic, regular of global dimension 3 and satisfies the Cohen-Macaulay property with Hilbert series

off the quadrics in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ determined by the $M_i$ have no common point and form a normalizing quadric system. In this case, $A$ is nonisomorphic and has no zero divisors.

Previous Known Results

Quadratic regular algebras of global dimension 1 [CV]

$K[x]$ where $\mu = 1$ and $M_1 = 1$ is a graded skew Clifford algebra.

Quadratic regular algebras of global dimension 2 [CV]

(i) $K[x_1, x_2]$ where $\lambda \in K^*$

$M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, where $\mu_{12} = \lambda$, is a graded skew Clifford algebra.

(ii) $K[x_1, x_2, x_3]$ where $\lambda_1 = \lambda_2 = 1$.

$M_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & \lambda_1 \\ 0 & \lambda_2 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & \lambda_1 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

For different choices $\lambda_i$ these algebras are regular graded skew Clifford algebras, and the point varieties that can be obtained are: $\mathbb{P}^2$, $\mathbb{P}^2 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^1$, where the latter is a nodal cubic.

Quadratic regular algebras of global dimension 3 [NVZ]

(iii) $A = K[x_1, x_2, x_3, x_4] \setminus (x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4) \setminus (x_1 x_2 x_3, x_2 x_3 x_4, x_3 x_4 x_1, x_4 x_1 x_2, x_1 x_2 x_3, x_2 x_3 x_4, x_3 x_4 x_1, x_4 x_1 x_2)$.

$r_1 = x_1 x_2 x_3 x_4, r_2 = x_1 x_2 x_3 x_4 - \lambda_1 x_1 x_2 x_3 x_4, r_3 = x_1 x_2 x_3 x_4 - \lambda_2 x_1 x_2 x_3 x_4, r_4 = x_1 x_2 x_3 x_4 - \lambda_3 x_1 x_2 x_3 x_4$, where $\lambda_1, \lambda_2 \in K$ for all $i$.

$M_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 \\ 0 & \lambda_1 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & \lambda_1 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

For different choices $\lambda_i$, these algebras are regular graded skew Clifford algebras, and the point varieties that can be obtained are: $\mathbb{P}^2$, $\mathbb{P}^2 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^1$, where the latter is a nodal cubic.

Future work

To find how many quadratic regular algebras of global 4 are graded skew Clifford algebras or twists of $\mathbb{P}^2$.

References


